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This paper describes A General Theory of filtering and control problems for linear systems, arising in particular from distributed parameter systems.

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Stochastic Filtering and Control of Linear Systems: A General Theory

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A large class of filtering and control problems for linear systems can be described as follows. We have an observed (stochastic) process $y(t)$ (say, an $m \times 1$ vector), t representing continuous time, $0 < t < T < \infty$. This process has the structure:

$$y(t) = v(t) + n_0(t)$$

where $n_0(t)$ is the unavoidable measurement error modelled as a white Gaussian noise process of known spectral density matrix, taken as the Identity matrix for simplicity of notation. The output $v(t)$ is composed of two parts: the response to the control input $u(t)$ and a random 'disturbance' $n_L(t)$ (sometimes referred to as 'load disturbance' or 'stale noise') also modelled as stationary Gaussian; we also assume the system responding to the control is linear and time-invariant so that we have:

$$v(t) = \int_0^t B(t-s)\, u(s)ds + n_L(t)$$

where $u(\cdot)$ is always assumed to be locally square integrable, and

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where $B(\cdot)$ is a 'rectangular' matrix function and $\int_0^\infty ||B(t)||^2 dt < \infty$.

We assume further more that the random disturbance is 'physically realizable' so that we can exploit the representation:

$$n_L(t) = \int_0^t F(t-\rho)\ N(\rho)\ d\rho$$

where $F(\rho)$ is a rectangular matrix such that

$$\int_0^\infty ||F(s)||^2 ds < \infty$$

where, in the usual notation,

$$||A||^2 = \text{Tr.}\ \ AA^*.$$

We assume that the process $n_L(t)$ is independent of the observation noise process $n_0(t)$.

It is more convenient now to rewrite the total representation as:

$$\left.\begin{aligned} y(t,\omega) &= v(t,\omega) + G\omega(t) \\ v(t,\omega) &= \int_0^t B(t-s)\ u(s)ds + \int_0^t \mathscr{F}(t-s)\ \omega(s)ds \end{aligned}\right\} \qquad (1.1)$$

where

$$GG^* = I$$

$$\mathscr{F}(t)G^* = 0$$

$\omega(\cdot)$ is white noise process in the appropriate product Euclidean space, and

$$\int_0^\infty ||\mathscr{F}(t)||^2 dt \ <\infty$$

We hasten to point out that we may replace the white noise formalism by a 'Wiener process' formalism for the above as:

$$Y(t,\omega) = \int_0^t v(s,\omega)ds + G\,W(t,\omega)$$

$$v(t,\omega) = \int_0^t B(t-s)u(s)ds + \int_0^t \mathscr{F}(t-s)dW(s,\omega)$$

It makes no difference to the theory that follows as to which formalism is used. The optimization problem we shall consider is a stochastic control ("regulator") problem in which the filtering problem is implicit: to minimize the effect of the disturbance on the output (or some components of it). More specifically, we wish to minimize:

$$\begin{aligned} &E \int_0^t [Qv(t,\omega),\ Qv(t,\omega)]dt \\ &+ E \int_0^t [u(t,\omega),\ u(t,\omega)]dt \end{aligned} \qquad (1.2)$$

E denoting expectation, where for each t, $u(t,\omega)$ must 'depend' only upon the available observation up to time t. We can show [1] that under the representation (1.1), (1.2), the optimal control may be sought in the class of 'linear' controls of the form:

$$u(t,\omega) = \int_0^t K(t,s)dY(s,\omega)ds$$

in the Wiener process formalism, or

$$\int_0^t K(t,s)\ y(s,\omega)ds \qquad (1.3)$$

in the white noise formalism.

This problem embraces already all the stochastic control problems for systems governed by ordinary differential equations by taking the special case where the Laplace transforms of $B(\cdot)$ and $\mathcal{F}(\cdot)$ are rational. But it also includes a wide variety of problems involving partial differential equations where the observation process $Y(t)$ for each t has its range in a finite dimensional Euclidean space (measurements at a finite number of points in the domain or on the boundary for example). One may argue that any physical measurement must be finite dimensional; in any case, the extension to the infinite dimensional case brings little that is new, and we shall not go into it here.

As a simple example of a non-rational case we may mention:

$$F(t) = t^{-3/2} e^{-1/t} \tag{1.4}$$

arising from boundary input in a half-infinite rod [5]. Note that the associated process $n_L(t)$ is not 'Markovian' even in the extended sense [2].

To solve our problem, our basic technique is to create an 'artificial' state space representation for (1.1). It is artificial in the sense that it has nothing to do with the actual state space that originates with the problems. We shall illustrate this with a specific example below. Without going into the system theoretic aspects involved, let us simply note that the controllable part of the original state space can be put in one-to-one correspondence with the controllable part of the artificial state space.

Let H denote $L_2[0,\infty;R_m]$ where m is the dimension of the observation process. Let A denote the operator with domain in H:

$$\mathcal{D}(A) = [f\varepsilon H \mid f(\cdot) \text{ is absolutely continuous with derivative}$$

$$f^1(\cdot) \;\varepsilon\; H \quad \text{also}],$$

and

$$Af = f^1$$

Let B denote the operator mapping the Euclidean space in which the controls range, into H by:

$$B\,u(t) \sim B(\zeta)u(t) \quad , 0 < \zeta < \infty$$

and similarly

$$\mathcal{F}\omega(t) \sim \mathcal{F}(\zeta)\omega(t) \quad 0 < \zeta < \infty$$

Assume now that $F(t)$ and $B(t)$ are 'locally' continuous, in $0 \leq t < \infty$. Then we claim that (1.1) is representable as (a partial differential equation!)

$$\left.\begin{aligned} \dot{x}(t) &= A\,x(t) + Bu(t) + \mathcal{F}\omega(t) \;;\; x(0) = 0. \\ y(t) &= C\,x(t) + G\omega(t) \end{aligned}\right\} \qquad (1.5)$$

(or appropriate 'Wiener-process' version), where C is the operator defined by:

$$\text{Domain of } C = [f\varepsilon H \mid f(t) \text{ is continuous in } 0 \leq t < \infty]$$

[or, $f(\cdot)$ is 'locally' continuous] and

$$C f = f(0)$$

[value at the origin of the 'continuous function' representative of $f(\cdot)$].

We can readily show that $x(t)$ is in the domain of C because of the assumption of local continuity. On the other hand we do not need to make the 'exponential rate of growth' assumptions as in the earlier version of the representation [3]. To see this we have only to note that (1.5) has the solution. (assuming that $u(\cdot)$ is locally square integrable):

$$x(t) = \int_0^t S(t-\sigma)Bu(\sigma)d\sigma + \int_0^t S(t-\sigma)\ \mathscr{F}\omega(\sigma)d\sigma \tag{1.5}$$

where $S(t)$ is the semigroup generated by A. Now

$h(t) = \int_0^t S(t-\sigma)\ Bu(\sigma)d\sigma$ is the function:

$$h(t,\zeta) = \int_0^t B(\zeta + t-\sigma)\ u(\sigma)d\sigma \qquad 0 < \zeta < \infty$$

and $h(t,\zeta)$ is locally continuous in $0 \leq \zeta < \infty$, because of the local continuity of $B(\cdot)$. Hence $h(t)$ is in the domain of C, for each t. Moreover

$$C\,h(t) = \int_0^t B\ (t-\sigma)\ u(\sigma)d\sigma$$

Similarly

$$C\int_0^t S(t-\sigma)\ \mathscr{F}\omega(\sigma)d\sigma = \int_0^t \mathscr{F}(t-\sigma)\omega(\sigma)d\sigma$$

which suffices to prove the representation. Of course to complete the representation we have that the cost functional (1.2) can be written:

$$E\int_0^t [QCx(t),\ QCx(t)]dt + E\int_0^t [u(t),\ u(t)]dt \tag{1.6}$$

In this form we have a stochastic control problem in a Hilbert Space, and we may apply the techniques of [4]; except for the complication that C is now unbounded, uncloseable. The 'operators' B and $\mathscr{F}$ are Hilbert-Schmidt and in this sense there is a simplification.

Even though C is uncloseable, let us note that

$$Cx(t) = \int_0^t B(t-\sigma)\ u(\sigma)d\sigma + \int_0^t \mathscr{F}(t-\sigma)\omega(\sigma)d\sigma$$

and hence is actually locally continuous in $0 \leq t$, and

$$g(\rho) = \int_0^\rho C\ S(\rho-\sigma)\ \ \mathscr{F}f(\sigma)d\sigma \qquad 0 < \rho < t$$

defines a linear bounded transformation on

$$W_n(t) = L_2\left((0,t),\ R_n\right)$$

where R_n is the Euclidean space in which $\omega(t)$ ranges, into

$$W_0(t) = L_2\left((0,t);R_m\right)$$

for each $0 < t$. We shall only consider $u(t)$ such that

$$u(t) = \int_0^t L\,(t,s)\; y(s)ds \qquad 0 < t < T \tag{1.7}$$

where

$$g(t) = \int_0^t L\,(t,s)\; f(s)ds \qquad 0 < t < T$$

defines a Hilbert-Schmidt operator mapping $W_0(T)$ into $W_C(T)$ where

$$W_C(T) = L_2[(0,T);\; R_p]$$

where R_p is the real Euclidean space in which $u(t)$ ranges for every t. The Hilbert-Schmidtness implies that $L(t,s)$ is Hilbert-Schmidt also a. e. and that

$$\int_0^T \int_0^t ||L(t,s)||^2_{H\cdot S} \; dt \quad < \infty$$

It is not difficult to see that

$$u(t) = \int_0^t L(t,s)y(s)ds$$

$$x(t) = \int_0^t S(t-\sigma)\; B\; u(\sigma)d\sigma + \int_0^t S(t-\sigma)\; \mathscr{F}\omega(\sigma)d\sigma$$

$$y(t) = C\; x(t) + G\; \omega(t)$$

defines $x(\cdot)$ uniquely, for each $\omega(\cdot)$.

2. The Filtering Problem.

Let us first consider the filtering problem for (1.1) taking $u(\cdot)$ to be identically zero. We shall see that this is an essential step in solving the control problem. Thus let, in the notation of Section 1,

$$\left.\begin{aligned} x(t,\omega) &= \int_0^t S(t-\sigma)\ \mathscr{F}\omega(\sigma)d\sigma \\ y(t,\omega) &= Cx(t,\omega) + G\omega(t) \end{aligned}\right\} \quad (2.1)$$

As we have noted earlier, the only difference from the standard problem treated in [4] is that C is uncloseable. Nevertheless since

$$Cx(t,\omega) = \int_0^t \mathscr{F}(t-\sigma)\ \omega(\sigma)d\sigma$$

and is continuous in t for each ω, we note that, denoting by $y_t(\omega)$ the element in $W_0(t)$ defined by

$$y(s,\omega), \quad 0 < s < t$$

we see that $y_t(\omega)$ is a weak Gaussian random variable with finite second moment in $W_0(t)$ for each t. Moreover y_t has the covariance operator:

$$I + L(t)\ L(t)^*$$

where $L(t)$ is defined by

$$L(t)f = g\ ; \quad g(\rho) = \int_0^\rho \mathscr{F}(\rho-\sigma)\ f(\sigma)d\sigma \quad 0 < \rho < t,$$

and is linear bounded on $W_n(t)$ into $W_0(t)$; and I is the identity operator on $W_0(t)$. Let

$$\hat{x}(t,\omega) = E\,[x(t,\omega) \mid y_t(\omega)]$$

Then $\hat{x}(t,\omega)$ belongs to the domain of C for each t and each ω and further

$$C\,\hat{x}(t,\omega) = E\,[Cx(t,\omega) \mid y_t(\omega)] \tag{2.2}$$

the novelty in this relation arising from the fact that C is unbounded. This can be seen readily as follows. We note that (see [4])

$$\hat{x}(t,\omega) = E\,[x(t,\omega)\, y_t(\omega)^*]\ [I + L(t)\, L(t)^*]^{-1}\ y_t(\omega) \tag{2.3}$$

where

$$E\,[x(t,\omega)\, y_t(\omega)^*]\, f = \int_0^t K(t,s) f(s) ds$$

where

$$K(t,\rho) = S(t-\rho) \int_0^\rho S(\rho-\sigma)\mathscr{F}\ \ \mathscr{F}(\rho-\sigma)^*\, d\sigma$$

and the corresponding element in H is given by

$$\int_0^t \int_0^\rho \mathscr{F}(t-\rho+\zeta)\ \mathscr{F}(\rho-\sigma)^* d\sigma\ \ f(s)ds, \quad 0 < \zeta < \infty$$

and is locally continuous in $0 \leq \zeta$, for any $f(\cdot)$ in $W_0(t)$. Hence it follows that $\hat{x}(t,\omega)$ is in the domain of C for each t and ω and further a simple verification establishes (2.2) since the right side of (2.2) is given by

$$E\,[Cx(t,\omega)\; y_t(\omega)^*]\;\; [I + L(t)\; L(t)^*]^{-1}$$

and for any f in $W_0(t)$:

$$E\,[Cx(t,\omega)\; y_t(\omega)^*]\, f = \; C\, E\,[x(t,\omega)\; y_t(\omega)^*]\, f$$

Relation (2.2) enables us to extend the arguments in [4] to show that

$$z(t,\omega) = y(t,\omega) - C\hat{x}(t,\omega) \qquad 0 < t < T$$

is again white noise. Let $P_f(t)$ denote

$$E\left[\Big(x(t,\omega) - \hat{x}(t,\omega)\Big)\Big(x(t,\omega) - \hat{x}(t,\omega)\Big)^*\right].$$

Then $P_f(t) = E\,[x(t,\omega)\; x(t,\omega)^*] - E\,[\hat{x}(t,\omega)\; \hat{x}(t,\omega)^*]$ and it follows that $P_f(t)$ maps into the domain of C. The covariance operator of $y(\cdot)$ as an element of $W_0(T)$ has the form

$$(I + R)$$

where R is Hilbert-Schmidt and hence the Krein factorization theorem (the Kernels being strongly continuous) as in [4] yields

$$(I+R)^{-1} = (I-\mathscr{L})^{*} (I-\mathscr{L})$$

where $\mathscr{L}$ is volterra and

$$z(\cdot,\omega) = (I-\mathscr{L})\, y(\cdot,\omega)$$

Moreover

$$(I-\mathscr{L})^{-1} = \dot{I} + M$$

where M is Hilbert-Schmidt also. Hence we can write

$$\hat{x}(\cdot,\omega) = Tz(\cdot,\omega)$$

where

$$Tf = g; \quad g(t) = \int_0^t J(t,\sigma)\, z(\sigma,\omega)d\sigma$$

and following [4] we must have that

$$J(t,\sigma) = S(t-\sigma)\, (C\, P_f(\sigma))^{*} \tag{2.4}$$

so that

$$P_f(t)x = \int_0^t S(\sigma)\mathscr{F}\,\mathscr{F}^{*}S(\sigma)^{*}x d\sigma - \int_0^t S(t-\sigma)(C\, P_f(\sigma))^{*}(C\, P_f(\sigma))S^{*}(t-\sigma)d\sigma$$

and in turn we have that, for x and y in the doman of A^{*}

$$[\dot{P}_f(t)x,y] = [P_f(t)x,A^*y] + [P_f(t)y, A^*x]$$

$$+ [\mathscr{F}x, \mathscr{F}y] - [C P_f(t) x, C P_f(t)y]; \tag{2.5}$$

$$P_f(0) = 0.$$

Further we have:

$$\hat{x}(t,\omega) = \int_0^t S(t-\sigma)\ (C P_f(\sigma))^*\ (y(\sigma,\omega) - C\hat{x}(\sigma,\omega))\ d\sigma$$

$$= -\int_0^t S(t-\sigma)\ (C P_f(\sigma))^*\ C\ \hat{x}(\sigma,\omega)\ d\sigma$$

$$+ \int_0^t S(t-\sigma)\ (C P_f(\sigma))^* y(\sigma,\omega)\ d\sigma \tag{2.6}$$

This is an 'integral equation' that $\hat{x}(t,\omega)$ satisfies. Moreover (2.6) has a unique solution. For suppose there were two solutions $\hat{x}_1(t,\omega)$, $\hat{x}_2(t,\omega)$. The difference, say $h(t)$, (fixing the ω), would satisfy

$$h(t) = -\int_0^t S(t-\sigma)\ (C P_f(\sigma))^* C\ h(\sigma)\ d\sigma$$

and hence we can deduce that:

$$C\ h(t) = -\int_0^t C\ S(t-\sigma)\ (C P_f(\sigma))^* (C\ h(\sigma))\ d\sigma$$

But $C\ h(\cdot)$ is an element of $L_2(0,T)$ and the right-side defines a Hilbert-Schmidt Volterra transformation which is then quasinilpotent. Hence $C\ h(\cdot)$ must be zero. Hence

$$C\,\hat{x}_1(t,\omega) = C\,\hat{x}_2(t,\omega)$$

Hence $z(t,\omega)$ remains the same:

$$z(t,\omega) = y(t,\omega) - C\,\hat{x}_1(t,\omega) = y(t,\omega) - C\,\hat{x}_2(t,\omega)$$

But

$$\hat{x}(t,\omega) = \int_0^t J(t,\sigma)\, z(\sigma,\omega)d\sigma$$

proving the uniqueness of solution of (2.6). We could also have deduced this from the uniqueness of the Krein factorization. We can also rewrite (2.6) in the differential form in the usual sense (see [4]):

$$\dot{\hat{x}}(t,\omega) = A\hat{x}(t,\omega) + (C\,P_f(t))^*(y(t,\omega) - C\,\hat{x}(t,\omega))$$

$$\hat{x}(0,\omega) = 0$$

yielding thus a generalization of the Kalman filter equations. Let us note in passing here that

$$A - (C\,P_f(t))^*C$$

is closed on the domain of A and the resolvent set includes the open right half plane. It does <u>not</u> however generate a <u>contraction</u> semigroup for $t > 0$.

The proof of uniqueness of solution to (2.5) can be given by invoking the dual control problem analogous to the case where C is bounded, as in [4] but will be omitted here because of limitation of space. From this it will also follow that $[P_f(t)x,x]$ is monotone in t.

Let C_n be defined on H by:

$$C_n f = g; \quad g(t) = n \int_0^{1/n} f(s)ds.$$

Then C_n is bounded. Hence it follows that

$$E\,(C_n\, x(t,\omega))\,(C_n\, x(t,\omega)^*)$$

$$= \int_0^t (C_n\, S(\sigma)\mathscr{F})\,(C_n\, S(\sigma)\mathscr{F})^* d\sigma,$$

and as n goes to infinite, the left side converges strongly and the right side yields

$$C\,(C\,R(t,f))^*\ ; \quad R(t,t) = E\,[x(t,\omega)\; x(t,\omega)^*].$$

In a similar manner we can show that

$$E\left[\left(C\,\hat{x}(t,\omega)\right)\left(C\,\hat{x}(t,\omega)\right)^*\right] = C\left(C\,\hat{R}(t,t)\right)^*\ ;$$
$$E\,[\hat{x}(t,\omega)\;\hat{x}(t,\omega)^*] = \hat{R}\,(t,t)$$

$$E\left[\left(C\,(x\,(t,\omega) - \hat{x}(t,\omega)\right)\ \left(C\,x(t,\omega) - C\,\hat{x}(t,\omega)\right)^*\right]$$
$$= C\,(C\,P_f(t))^*$$

We are of course most interested in the case $T \to \infty$. We have seen that $[P_f(t)\,x,x]$ is monotone. Also

$$[P_f(t)x,x] \leq [R\,(t,t)x,x] = \int_0^t [S(\sigma)\mathscr{F}\,\mathscr{F}^* S(\sigma)^* x,x]d\sigma$$

Let us assume now that

$$\int_0^\infty ||\mathscr{F}^* S(\sigma)^* x||^2 d\sigma = [R_\infty x, x] < \infty \tag{2.7}$$

(This is clearly satisfied in our example (1.4)).

Then $P_f(t)$ also converges strongly, to P_∞, say; further P_∞ maps into the domain of C and satisfies

$$P_\infty = R_\infty - . \int_0^\infty S(\sigma)\ (C\ P_\infty)^* (C\ P_\infty)\ S(\sigma)^* d\sigma$$

and hence also the algebraic equation:

$$0 = [P_\infty x, A^* y] + (P_\infty y, A^* x] + [\mathscr{F}^* x, \mathscr{F}^* y] - [\ C\ P_\infty x, C\ P_\infty y] \tag{2.8}$$

which has a unique solution.

3. The Control Problem.

Because of space limitations, we shall have to limit the presentation to the main results, emphasing only the differences arising due to the unboundedness of C. Thus, defining as in [4, Chapter 6], and confining ourselves to controls defined by (1.7);

$$x(t,\omega) - x_u(t,\omega) = \tilde{x}(t,\omega)$$
$$C\ \tilde{x}(t,\omega) + G\omega(t) = \tilde{y}(t,\omega)$$

where

$$\dot{x}_u(t,\omega) = A\ x(t,\omega) + B\ u(t,\omega)$$

we can invoke the results of section 2 to obtain that

$$z(t,\omega) = \tilde{y}(t,\omega) - C\,\hat{\tilde{x}}(t,\omega)$$

where

$$\hat{\tilde{x}}(t,\omega) = E\,[\tilde{x}(t,\omega) \mid \tilde{y}(\rho,\omega),\ 0 < \rho < t]$$

yields white noise. We can then also proceed as in [4] to show that we can also express any $u(t,\omega)$ satisfying (1.7), also as

$$u(t,\omega) = \int_0^t m(t,\rho)\ z(\rho,\omega)d\rho$$

where the operator is Hilbert-Schmidt. The separation theorem follows easily from this, and we can show that the optimal control is given by

$$u_0(t,\omega) = -\int_t^T (Q\ C\ S(\rho-t)B)^*\ \hat{x}(\rho,\omega)d\rho \tag{3.1}$$

where

$$\hat{x}\ (\rho,\omega) = \hat{\tilde{x}}\ (\rho,\omega) + x_u(\rho,\omega)$$

and hence as in section 2, is the unique solution of

$$\dot{\hat{x}}\ (\rho,\omega) = A\ \hat{x}(\rho,\omega) + B\ u_0(\rho,\omega) + \left(C\ P_f(\rho)\right)^*\left(y(\rho,\omega) - C\ \hat{x}(\rho,\omega)\right)$$

$$\hat{x}\ (0,\omega) = 0$$

Further we can follow [4], making appropriate modifications of the unboundedness of C, to deduce from (3.1) that

$$u_0(t,\omega) = -(P_c(t)B)^* \hat{x}(t,\omega) \tag{3.2}$$

where $P_c(t)$ is the solution of

$$\begin{aligned}[\dot{P}_c(t)x,y] &= [P_c(t)x,Ay] + [P_c(t)Ax,y] \\ &\quad + [QCx, QCy] \\ &\quad - [(P_c(t)B)^*x, (P_c(t)B)^*y]; \\ P_c(T) &= 0 \end{aligned} \tag{3.3}$$

for x,y in the domain of A.

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